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ON THE NUMBER OF LATTICE POINTS IN CONVEX SYMMETRIC BODIES AND THEIR DUALS

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Let $K \subset \mathbf{R}^n$ be a convex centrally symmetric bounded and absorbing set, $\#(K \cap \mathbf{Z}^n)$ the number of lattice points it contains, $\mathrm{vol}(K)$ its standard volume, and K^* its polar convex set. We prove below that the quantity

$$\#(K \cap \mathbf{Z}^n)/\big(\#(K^* \cap \mathbf{Z}^n)\cdot \mathrm{vol}(K)\big)$$

is bounded, above and below, by positive constants, depending on n but not on K (Theorem 1).

This result may be seen as an arithmetic analog of the Riemann–Roch theorem for a bundle E on a smooth projective curve C: K (resp. K^*, $\log(\#(K \cap \mathbf{Z}^n))$, $\log \mathrm{vol}(K)$) plays the role of E (resp. $\Omega^1 \otimes E^*$, $h^0(C,E)$, $\deg(E)$). One knows that Minkowski's theorem stating that $K \cap \mathbf{Z}^n \neq \{0\}$ when $\mathrm{vol}(K)$ is big enough is analogous to the theorem of Riemann that E has a nontrivial section when $\deg(E)$ is big enough. Our result completes this analogy. We also deduce several corollaries of this theorem.

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1. The main result

1.1. Let K be a convex centrally symmetric bounded and absorbing set in $\mathbf{R}^n$. We consider two invariants of K, the number of lattice points in K

$$M(K) = \#(K \cap \mathbf{Z}^n) \tag{1}$$

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and the volume of K

$$V(K) = \mathrm{vol}(K) \tag{2}$$

for the standard measure on $\mathbf{R}^n$. When $n = 0$ we take $M(K) = V(K) = 1$.

Let $\langle x, y\rangle = \sum_i x_i y_i$ be the usual inner product on $\mathbf{R}^n$ and

$$K^* = \{x \in \mathbf{R}^n;\ \sup_{y \in K} \langle x, y\rangle \leq 1\} \tag{3}$$

the polar body of K.

According to Bourgain and Milman [2] there is an absolute constant $c > 0$ such that

$$V(K)V(K^*) \geq c^n V(B_n)^2, \tag{4}$$

where B_n is the standard unit ball and

$$V(B_n) = 2\pi^{n/2}\Big/\Gamma\left(\frac{n}{2} + 1\right).$$

THEOREM 1. *For any K as above*

$$6^{-n} \leq \frac{M(K)}{M(K^*)V(K)} \leq 6^n c^{-n} V(B_n)^2.$$

1.2. To prove Theroem 1 it is enough to get the lower bound

$$6^{-n} \leq M(K)/(M(K^*)V(K)). \tag{5}$$

Indeed, assume (5) holds for all K. Applying this inequality to K^* we get

$$M(K^{**}) \leq 6^n M(K^*)V(K^*)^{-1}.$$

But $K \subset K^{**}$ so $M(K) \leq M(K^{**})$. Using (4) we conclude that

$$M(K) \leq M(K^*)V(K)6^n c^{-n} V(B_n)^{-2},$$

i.e. the upper bound in Theorem 1.

1.3. REMARKS. (a) When K is an ellipsoid we have

$$V(K)V(K^*) = V(B_n)^2,$$

therefore Theorem 1 holds with c replaced by 1.

(b) By the argument in 1.2, we see that Theorem 1 remains valid if we replace K^* by the set $K^\vee$ of $y \in \mathbf{R}^n$ such that, for all $x \in K$, $|\langle x,y\rangle| < 1$.

1.4. Lemma 2. *Let $I \subset \mathbf{R}$ be an interval. Then*

$$\#(I \subset \mathbf{Z}) \leq \sup\{\tfrac{3}{2}V(I),2\}.$$

Proof. Here $V(I)$ is the usual length of I. Clearly

$$\#(I \cap \mathbf{Z}) \leq V(I) + 1.$$

The conclusion follows from the fact that $\#(I \cap \mathbf{Z})$ is an integer.

1.5. Proposition 3. *Assume that the finite set $K \cap Z^n$ spans the lattice $\mathbf{Z}^n$. Then*

$$M(K)V(K^*) \leq 6^n.$$

Proof. We proceed by induction on n. When $n = 0$ this is true by convention. Assume the Proposition holds for $n - 1$. Let v be a nonzero primitive element in $K \cap \mathbf{Z}^n$. Define $\Lambda_0 = \mathbf{Z}v$, $\Lambda_1 = \Lambda/(\mathbf{Z}v)$, $E_0 = \mathbf{R}v$, $E_1 = \Lambda_1 \otimes_{\mathbf{Z}} \mathbf{R}$, $p : \mathbf{R}^n \to E_1$ the projection, $K_0 = K \cap E_0$ and $K_1 = p(K)$.

Since $p(K \cap \mathbf{Z}^n) \subset (K_1 \cap \Lambda_1)$ we get

$$M(K) = \#(K \cap \mathbf{Z}^n) \leq \#(K_1 \cap \Lambda_1) \times \operatorname*{Max}_{x \in E_1} \#\big(p^{-1}(x) \cap K \cap \mathbf{Z}^n\big). \tag{6}$$

Given any $y \in \mathbf{Z}^n$ and $x = p(y)$, the map

$$\varphi : \mathbf{R} \to p^{-1}(x)$$

mapping $\lambda \in \mathbf{R}$ to $\varphi(\lambda) = y + \lambda v$ is an isomorphism such that $\varphi(\mathbf{Z}) = p^{-1}(x) \cap \mathbf{Z}^n$ and $\varphi^{-1}(p^{-1}(x) \cap K)$ is an interval in $\mathbf{R}$. From Lemma 2 we get

$$\#\big(p^{-1}(x) \cap K \cap \mathbf{Z}^n\big) \leq \sup_{x \in E_1} \{\tfrac{3}{2}V\big(p^{-1}(x) \cap K\big),2\}. \tag{7}$$

The set $p^{-1}(0)K \cap \mathbf{Z}^n$ contains more than two elements and, by the Brunn-Minkowski theorem,

$$V(p^{-1}(x) \cap K) \leq V(p^{-1}(0) \cap K) = V(K_0). \tag{8}$$

Therefore, by (6), (7) and (8),

$$M(K) \leq M(K_1)\tfrac{3}{2}V(K_0).$$

We may now apply the induction hypothesis to K_1 and the lattice Λ_1 of rank $n-1$. We get

$$M(K) \leq 6^{n-1}\tfrac{3}{2}\,V(K_1^*)^{-1}V(K_0). \tag{9}$$

Since E_0 has dimension one, we have

$$V(K_0)V(K_0^*) = 4. \tag{10}$$

On the other hand, again by the Brunn-Minkowski theorem (see for example [2], (3.1)),

$$V(K^*) \leq V(K_1^*)V(K_0^*). \tag{11}$$

Using (9), (10) and (11) we conclude that

$$M(K) \leq 6^{-n}V(K^*)^{-1}. \qquad \text{q.e.d.}$$

1.6. Let us now consider a general $K \subset \mathbf{R}^n$. Let $\Lambda_0 \subset \mathbf{Z}^n$ be the lattice generated by $K^* \cap \mathbf{Z}^n$, n_0 its rank, and $E_0 = \Lambda_0 \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}^{n_0}$. Clearly

$$M(K^*) = M(K^* \cap E_0) \tag{12}$$

and, by applying Proposition 3 to $\Lambda_0 \subset E_0$, we get

$$M(K^* \cap E_0) \leq V\big((K^* \cap E_0)^*\big)^{-1}6^{n_0}. \tag{13}$$

Let $\mathring{K} = (K^*)^\vee \subset K$ be the set of $x \in K$ such that, for all $y \in K^*$,

$$|\langle x,y\rangle| < 1.$$

If $x \in \mathring{K} \cap \mathbf{Z}^n$ and $y \in K^* \cap \mathbf{Z}^n$, the integer $\langle x,y\rangle$ must vanish. Therefore x lies in the orthogonal subspace $E_0^\perp$ to E_0. We conclude that

$$M(K) \geq M(\mathring{K}) = M(\mathring{K} \cap E_0^\perp). \tag{14}$$

According to Van der Corput's variant of the Minkowski theorem ([5], II.7.2, Theorem 1 or [3], III.2.2, Theorem 2), for any $\epsilon > 0$, $M(\mathring{K} \cap E_0^\perp) \geq 1 + 2^{1+n_0-n} \times [V(\mathring{K} \cap E_0^\perp) - \epsilon]$. In particular

$$M(\mathring{K} \cap E_0^\perp) \geq 6^{n_0-n}V(\mathring{K} \cap E_0^\perp). \tag{15}$$

Combining (12), (13), (14) and (15) we get

$$M(K)/M(K^*) \geq 6^{-n}V(\mathring{K} \cap E_0^\perp)V\big((K^* \cap E_0)^*\big).$$

But $V(\mathring{K} \cap E_0^\perp) = V(K \cap E_0^\perp)$ and the closure of $(K^* \cap E_0)^*$ is the orthogonal projection to E_0 of the closure of K. Therefore, by Brunn–Minkowski (see [2], (3.1)),

$$V(\mathring{K} \cap E_0^\perp)V\big((K^* \cap E_0)^*\big) \geq V(K).$$

We conclude that

$$M(K)/M(K^*) \geq 6^{-n}V(K),$$

i.e. (5) holds and Theorem 1 follows.

1.7. Notice that, when $K \cap \mathbf{Z}^n$ spans the lattice $\mathbf{Z}^n$, $M(K^\vee) = 1$. Therefore, replacing K by $K^\vee$, we see that Proposition 3 is a special case of Theorem 1.

2. Some consequences

2.1. *Scaling.* For any positive real number λ denote by λK the image of K by multiplication by λ in $\mathbf{R}^n$.

PROPOSITION 4. *If* $\lambda > 1$, *then*

$$M(K) \leq M(\lambda K) \leq \lambda^n M(K)(36)^n c^{-n} V(B_n)^{-2}.$$

PROOF. The first inequality is clear since $K \subset \lambda K$. On the other hand $(\lambda K)^*$ is contained in K^*, so, by applying Theorem 1 twice,

$$\begin{aligned} M(\lambda K) &\leq M((\lambda K)^*)V(\lambda K)6^n c^{-n} V(B_n)^{-2} \\ &\leq M(K^*)V(K)\lambda^n 6^n c^{-n} V(B_n)^{-2} \\ &\leq M(K)6^n \lambda^n 6^n c^{-n} V(B_n)^{-2}. \end{aligned}$$

q.e.d.

2.2. *Blichfledt numbers.* Let

$$B(K) = \operatorname*{Max}_{x \in \mathbf{R}^n} \#\{K \cap (x + \mathbf{Z}^n)\}.$$

PROPOSITION 5.

$$M(K) \leq B(K) \leq M(K)(72)^n c^{-n} V(B_n)^{-2}.$$

PROOF. Since K is convex and centrally symmetric we have

$$M(K) \leq B(K) \leq M(2K).$$

We can then apply Proposition 4.

2.3. *Successive minima*

2.3.1. When $i = 1, \ldots, n$ let

$$\lambda_i = \operatorname{Inf}\{\lambda > 0/\operatorname{rk}(\lambda K \cap \mathbf{Z}^n) \geq i\}.$$

Assume $\lambda_k < 1$ and $\lambda_{k+1} \geq 1$ (or $k = n$).

PROPOSITION 6. *The quotient $\lambda_1 \cdots \lambda_k / M(K)$ is bounded above and below by constants independent of K.*

PROOF. Let $\Lambda_0 \subset \mathbf{Z}^n$ be the lattice generated by $K \cap \mathbf{Z}^n$ and $K_0 = K \cap (\Lambda_0 \otimes_{\mathbf{Z}} \mathbf{R})$. Since Λ_0 is generated by K_0, we have $M(K_0^\vee) = 1$. Therefore, by Theorem 1 and 1.3,

$$M(K) = M(K_0) \sim V(K_0)$$

(where $a(K) \sim b(K)$ means that a/b is bounded above and below independently of K). Since $\lambda_1, \ldots, \lambda_k$ are all the successive minima of Λ_0, we get, by Minkowski's theorem on successive minima ([5], II.9, Theorems 1 and 2),

$$\lambda_1 \cdots \lambda_k \sim V(K_0). \qquad \text{q.e.d.}$$

2.3.2. Let $\lambda_1^*, \ldots, \lambda_n^*$ be the successive minima of K^*. By a theorem of Mahler ([5], II.14.2, Theorem 5)

$$\lambda_i \lambda_{n-i}^* \sim 1.$$

This, together with $\lambda_1 \cdots \lambda_n \sim V(K)$, shows that Proposition 6 is essentially equivalent to Theorem 1.

Indeed, in the function field case, one may prove directly the analog of Proposition 6 [8] (it is an equality), and Armitage deduced from it the Riemann-Roch theorem on curves [1].

2.4. *Hermitian modules*

2.4.1. Let F be a number field, $\mathcal{O}$ its ring of integers, $d = [F, \mathbf{Q}]$ its degree, r_1 (resp. r_2) its number of real (resp. complex) places ($d = r_1 + 2r_2$), $|D_F|$ the absolute value of its discriminant over $\mathbf{Q}$, and Σ the set of complex imbeddings $\sigma : F \to \mathbf{C}$. A *Hermitian module* $\overline{M} = (M, h)$ over $\mathcal{O}$ is a finitely generated $\mathcal{O}$ module M equipped with a Hermitian scalar product h_σ on the complex vector space $M_\sigma =$

$M \otimes_A \mathbf{C}$ attached to every $\sigma \in \Sigma$. Furthermore, if $\bar{\sigma}$ is the complex conjugate of σ, $h_{\bar{\sigma}}$ should be the conjugate of h_σ.

Given $\overline{M}$ as above, we denote by $\| \, \|_\sigma$ the norm defined by h_σ on M_σ (i.e. $\|x\|_\sigma^2 = h_\sigma(x,x)$) and by $h^0(M) = h^0(\overline{M}) \in \mathbf{R}^+$ the logarithm of the number of elements $m \in M$ such that, for every $\sigma \in \Sigma$, $\|m\|_\sigma \leq 1$. Let $n = \dim_F(M \otimes_{\mathcal{O}} F)$ be the rank of M, and $M_{\mathrm{tors}} \subset M$ its torsion subgroup. The *degree* of $\overline{M}$, $\deg(M) = \deg(\overline{M}) \in \mathbf{R}$, is defined as follows [7]. First

$$\deg(M) = \log \#(M_{\mathrm{tors}}) + \deg(\Lambda^n(M/M_{\mathrm{tors}})) \tag{16}$$

where Λ^n is the n-th exterior power of projective $\mathcal{O}$-modules. Second, when $n = 1$, for any $m \in M/M_{\mathrm{tors}}$,

$$\deg(M) = \log \#(M/\mathcal{O}m) - \sum_{\sigma \in \Sigma} \log \|m\|_\sigma \tag{17}$$

(this number is independent of the choice of m).

The $\mathcal{O}$-module

$$\omega = \mathrm{Hom}_{\mathbf{Z}}(\mathcal{O},\mathbf{Z}), \tag{18}$$

where $x\varphi(y) = \varphi(xy)$ if $x,y \in \mathcal{O}$ and $\varphi \in \omega$, has rank one. It is generated, up to torsion, by the trace $\mathrm{Tr} : \mathcal{O} \to \mathbf{Z}$. As in [9] we fix a Hermitian metric on ω by deciding that $|\mathrm{Tr}|_\sigma = 1$ (resp. $|\mathrm{Tr}|_\sigma = 2$) if $\sigma = \bar{\sigma}$ (resp. $\sigma \neq \bar{\sigma}$).

Given $\overline{M}$, the $\mathcal{O}$-module $M^* = \mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O})$ is equipped with the dual metric, and $M^\vee = \omega \otimes_{\mathcal{O}} M^*$ with the tensor product of the metrics on ω and M^*.

THEOREM 2. *Given any Hermitian module $\overline{M}$ of rank n over $\mathcal{O}$, the following inequalities hold*:

$$-C(r_1,r_2,n) \leq h^0(M) - h^0(\omega \oplus_{\mathcal{O}} M^*) - \deg(M) - n\chi(\mathcal{O}) \leq C(r_1,r_2,n)$$

where

$$\chi(\mathcal{O}) = r_2 \log(2) - \tfrac{1}{2}\log|D_F|$$

and

$$C(r_1,r_2,n) = nd\log(3) + nr_1\log(2) - r_1\log(V(B_n)n!)$$
$$- r_2\log(V(B_{2n})(2n)!) + \log((nd)!).$$

2.4.2. To prove Theorem 2, we first notice that the number we want to estimate is unchanged when M gets replaced by M/M_{tors}. So we may assume that M is torsion free.

Let

$$M_{\mathbf{R}} \subset \bigoplus_{\sigma \in \Sigma} M_\sigma = M \otimes_{\mathbf{Z}} \mathbf{C}$$

be the real subspace fixed under complex conjugation, and

(19) $$\chi(M) = -\log \mu(M_{\mathbf{R}}/M),$$

where μ is the measure on $M_{\mathbf{R}}$ and $M_{\mathbf{R}}/M$ induced by the scalar products h_σ. If $\mathcal{O}$ has the trivial metric, we get

(20) $$\chi(M) = \deg(M) + n\chi(\mathcal{O})$$

([7],[9]). The map

$$\alpha : M^\vee = \omega \otimes_{\mathcal{O}} M^* \to \mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z})$$

sending $T \otimes u$ to the composition $u \circ T$ is an isomorphism of $\mathcal{O}$-modules. For any complex imbedding $\sigma : F \to \mathbf{C}$, the induced isomorphism

$$\alpha_\sigma : M_\sigma^\vee = \omega_\sigma \otimes_{\mathbf{C}} M_\sigma^* \to (\mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z}))_\sigma = M_\sigma^*$$

maps $v_\sigma = \mathrm{Tr} \otimes u_\sigma$ to u_σ (resp. $u_\sigma + \bar{u}_\sigma$) if $\sigma = \bar{\sigma}$ (resp. $\sigma \neq \bar{\sigma}$). Since $|\mathrm{Tr}|_\sigma = 1$ (resp. 2), α_σ is an isometry.

Denote by $K \subset M_{\mathbf{R}}$ the set of elements x such that $\|x\|_\sigma \leq 1$ for every $\sigma \in \Sigma$. Its dual in $\mathrm{Hom}_{\mathbf{Z}}(M_{\mathbf{R}},\mathbf{R})$ is isomorphic by α^{-1} to the set $K^* \subset M_{\mathbf{R}}^\vee$ of elements $(v_\sigma = \mathrm{Tr} \otimes u_\sigma)$ such that, for any $(x_\sigma) \in K$,

$$\left| \sum_{\sigma = \bar{\sigma}} u_\sigma(x_\sigma) + \sum_{\sigma \neq \bar{\sigma}} (u_\sigma + \bar{u}_\sigma)(x_\sigma) \right| \leq 1.$$

Therefore K^* is the set of $(v_\sigma) \in M_{\mathbf{R}}^\vee$ such that

$$\sum_{\sigma \in \Sigma} \|v_\sigma\| \leq 1.$$

If $V(K^*)$ is the volume of K^* for the Haar measure on $M_{\mathbf{R}}^\vee$ which gives $M^\vee$ covolume one, Theorem 1 applied to $(M^\vee,K^*)$ gives the inequality

(21) $$\log \#(M^\vee \cap K^*) - \log \#(M \cap K) - \log V(K^*) \geq -nd \log(6).$$

Notice that $\log \#(M \cap K) = h^0(M)$ and

(22) $$\log \#(M^\vee \cap K^*) \leq h^0(\omega \otimes_{\mathcal{O}} M^*).$$

If μ is the measure induced by the scalar product on $M_{\mathbf{R}}^{\vee}$, we have

$$\log V(K^*) = \chi(M^\vee) + \log \mu(K^*), \tag{23}$$

and $\mu(K^*)$ is the euclidean volume of the set of vectors $(x_i, y_j) \in (\mathbf{R}^n)^{r_1} \times (\mathbf{C}^n)^{r_2}$ such that

$$\sum_{i=1}^{r_1} |x_i| + 2 \sum_{j=1}^{r_2} |y_j| \leq 1.$$

One computes

$$\mu(K^*) = 2^{2r_2 n}(V(B_n)n!)^{r_1}(V(B_{2n})(2n)!)^{r_2}/((nd)!). \tag{24}$$

Using (20) we get

$$\begin{aligned} \chi(M^\vee) &= \deg(M^\vee) + n\chi(\mathcal{O}) \\ &= \deg(M^*) + n\deg(\omega) + n\chi(\mathcal{O}) \\ &= -\deg(M) - n\chi(\mathcal{O}), \end{aligned} \tag{25}$$

since

$$\begin{aligned} \deg(\omega) &= \log \#(\omega/\mathcal{O}.\mathrm{Tr}) - \sum_{\sigma \in \Sigma} \log|\mathrm{Tr}|_\sigma \\ &= \log|D_F| - 2r_2 \log(2) \\ &= -2\chi(\mathcal{O}) \end{aligned}$$

([6], V.2., Lemma 2; [9]).

From (21), (22), (23), (24) and (25) we conclude that

$$\begin{aligned} h^0(M) - h^0(\omega \otimes_{\mathcal{O}} M^*) - \deg(M) - n\chi(\mathcal{O}) &\leq -\log \mu(K^*) + nd\log(6) \\ &= C(r_1, r_2, n). \end{aligned} \tag{26}$$

If we replace M by $M^\vee$ in (26) and use (25), we obtain the lower bound in Theorem 2.

2.4.3. Remark. When $F = \mathbf{Q}$, Theorem 2 reads (as in 1.3)

$$6\log(n) + \log(V(B_n)) \leq h^0(M) - h^0(M^*) - \deg(M) \leq 6\log(n) - \log(V(B_n)). \tag{27}$$

Given F and M as in Theorem 2, we may apply (27) to M viewed as a module $M_{\mathbf{Z}}$ of rank nd over $\mathbf{Z}$. By definition, $\exp(h^0(M_{\mathbf{Z}}))$ is then the number of elements m in M such that

$$\sum_{\sigma\in\Sigma} \|m\|_\sigma^2 \leq 1.$$

2.5. *Exact sequences*

2.5.1. Let F be a number field as in 2.4. For any Hermitian module $\overline{M}$ we define

$$h^1(M) = h^0(\omega \otimes_{\mathcal{O}} M^*). \tag{28}$$

Let

$$0 \to M_1 \to M_2 \to M_3 \to 0 \tag{29}$$

be an exact sequence of finitely generated $\mathcal{O}$-modules, and $(h_{2,\sigma})$ a Hermitian metric on M_2. This metric induces metrics $(h_{1,\sigma})$ on M_1 and $(h_{3,\sigma})$ on M_3 (here we view $M_{3,\sigma}$ as the orthogonal complement to $M_{1,\sigma}$ in $M_{2,\sigma}$). We then say that we have an *exact sequence of Hermitian modules*

$$0 \to \overline{M}_1 \to \overline{M}_2 \to \overline{M}_3 \to 0.$$

We let n_i be the rank of M_i, $i = 1,2,3$.

Proposition 7. *For any exact sequence of Hermitian modules, the following inequalities hold*:

(i) $0 \leq h^0(M_1) \leq h^0(M_2) \leq h^0(M_1) + h^0(M_3)$,

(ii) $h^0(M_1) - h^0(M_2) + h^0(M_3) - h^1(M_1) \leq \sum_{i=1}^3 C(r_1, r_2, n_i)$.

2.5.2. The proof of (i) is easy from the definitions. To check (ii) we apply Theorem 2 to M_1, M_2 and M_3:

$$\begin{aligned} &h^0(M_1) - h^0(M_2) + h^0(M_3) - h^1(M_1) \\ &\quad \leq \deg(M_1) + n_1\chi(\mathcal{O}) - \deg(M_2) + n_2\chi(\mathcal{O}) + \deg(M_3) + n_3\chi(\mathcal{O}) \\ &\qquad + h^0(M_2^\vee) + h^0(M_3^\vee) + \sum_{i=1}^3 C(r_1, r_2, n_i). \end{aligned}$$

But the degree and the rank are additive on exact sequences. Furthermore $M_3^\vee \subset M_2^\vee$ and the metrics are compatible, therefore $h^0(M_3^\vee) \leq h^0(M_2^\vee)$. This proves Proposition 7.

2.5.3. In [4] it is proved that an inequality like (ii) holds, with the right-hand side replaced by zero, when $h^0(M)$ is replaced by the logarithm of the Blichfeldt number of the unit ball, and $M_{2,\mathbf{R}}$ is equipped with the sup norm of $M_{1,\mathbf{R}}$ and $M_{3,\mathbf{R}}$ (for some **R**-linear splitting of the exact sequence; see [4] for a more general hypothesis).

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